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Wave Propagation in Random Media

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Abstract

Wave propagation and random media are defined and the nature of the mathematical problems arising in wave propagation in random media is described. The two principal types of methods for solving these problems - honest and dishonest methods - are explained. These methods are first illustrated by considering the geometrical optics of a random medium by one method of each type. Some new results are obtained by an honest method and some errors in a previous work are pointed out. Comparison is made between the results of the two methods and the reasons why they disagree are explained. As a second illustration of an honest method, an analysis of the reduced wave equation in a random medium is presented. Some known results are obtained in a new way which is simpler than the usual one and which appears to be capable of yielding further results.

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1. Introduction

Wave propagation is one of the means by which energy travels. 'Propagation' is the process whereby the energy moves from one region of space to an adjacent region, and 'wave' is a general term for a moving spatial distribution of energy. The matter in the region of space through which the propagation occurs is called the transmission medium. A random or stochastic medium is a family of media together with a probability distribution which specifies the probabilities of the various members of the family. Thus wave propagation in a random medium refers to propagation in each member of the family of media, together with the probability of each member. This probability, when associated with the wave motion in each medium, characterizes a random wave motion.

Mathematically a wave motion is described by a vector-valued function $u(x,t)$ of the position vector x and the time t. As a consequence of the physical laws governing the wave motion, the function $u(x,t)$ satisfies certain equations. Usually they are partial differential equations of hyperbolic type and often of symmetric hyperbolic type. The transmission medium is characterized by a vector-valued function $n(x,t)$ which enter the coefficients of the equations. A wave propagation problem is that of determining a solution $u(x,t)$ of the equations which satisfies certain auxiliary conditions. These conditions are usually initial and boundary conditions. The problem is said to be well set, well posed or properly posed if it possesses a unique solution which depends continuously, in an appropriate norm, upon the data given in the auxiliary conditions and upon the coefficients in the equation.

A random medium is characterized by a family of functions $n(x,t,\omega)$ depending upon a parameter ω. Here ω is a point in a probability space Ω in

which a probability measure $dP(\omega)$ is defined. Of course the function $n(x,t,\omega)$ must be measurable with respect to $dP(\omega)$. If a well-posed wave propagation problem is formulated for the random medium, it has a solution $u(x,t,\omega)$ for each ω. This family of solutions describes a random wave motion. The equations satisfied by u with the coefficient $n(x,t,\omega)$ are called stochastic equations - stochastic differential equations if they are differential equations. Thus in this case wave propagation in a randum medium is part of the subject of stochastic differential equations.

The reasons for studying random wave propagation are exactly the same as those for studying any other random phenomenon. There are essentially two such reasons. In the first place, we may wish to consider a case in which the medium is not known precisely, but in which it is known to be a member of a certain family of media. If we also know the probability that it is any particular member of the family, we can determine the probability that the wave motion is any one of the associated family of wave motions. Then we can determine the expected or mean wave motion, as well as its variance and other statistics. This statistical information can be used to estimate what is likely to be the wave motion in the case under consideration. In particular some statistical properties of u may depend only upon known statistics of n. When we consider many media, all of which are members of the same family, we may expect the statistics of the corresponding wave motions to correspond to those of the random wave motion occurring in the random medium.

In the second place we may wish to consider a known medium with very complex properties. Then the associated wave motion will also be very complex and its precise determination may be impractical. Certain statistical properties of the wave motion in a random medium, of which the medium in

question is a member, may be less complex and more easily determinable. In other words, it may be easier to solve the random problem. If the random medium is appropriately chosen, these statistical properties may be closely related to the actual properties of the wave motion in the known medium under consideration.

We shall see that there are two kinds of methods used to solve problems of wave propagation in random media. These may be called 'honest' and 'dishonest' methods respectively. In an honest method the solution $u(x,t,\omega)$ is first determined for each value of ω. This solution may sometimes be found exactly and explicitly, but more often it is expressed in the form of a series in some parameter, or as a sequence of iterates, or by some other approximation procedure. In the process of solving for $u(x,t,\omega)$ randomness plays no role and therefore it provides no advantage. The second step in an honest method is to compute the mean value of $u(x,t,\omega)$, as well as its variance and other statistics, from the explicit expression. In this step randomness may have the helpful effect of yielding simpler expressions for the statistics of u than those for u itself.

In a dishonest method randomness is utilized before $u(x,t,\omega)$ is determined. For example, if the mean value $\langle u\rangle$ is sought, the original equations for u may be 'averaged' to yield equations for $\langle u\rangle$. In such cases the averaged equations also involve terms like $\langle u^2\rangle$ or $\langle nu\rangle$. Dishonesty enters through the assumption that $\langle u^2\rangle = \langle u\rangle^2$ or $\langle nu\rangle = \langle n\rangle\langle u\rangle$ or some similar unproved assumption. The reason for making such assumptions is to obtain determinate equations for $\langle u\rangle$. Other examples of dishonest methods will be described later. In all cases probability is introduced before u is determined and an unproved assumption is made about some statistical property of the random wave motion. The assumption usually simplifies the problem so

that it becomes solvable.

Dishonest methods have the advantage over honest ones that they simplify the problem to be solved. As a consequence, a problem which can be solved honestly only by a perturbation method might be solved dishonestly without the use of a perturbation expansion. Then the dishonest solution would be applicable for all values of the relevant parameter while the honest solution would be valid only for small values of it. For this and similar reasons, many of the significant and non-trivial results in the theory of wave propagation in random media have been obtained by dishonest methods. Many of these results have compared well with experiment. Thus one of the important mathematical problems in this field is to justify the dishonest methods by showing that their results, in some sense, are approximations to honest solutions. A clear understanding of the circumstances in which this is the case would permit the introduction of many more useful dishonest procedures.

Most of the work on this subject has been done since 1945, having been stimulated by practical problems of radio wave propagation through the atmosphere and ionosphere, sound wave propagation in the ocean and the atmosphere, light transmission through the atmosphere, etc. The recent book by L.A. Chernov [1] contains a rather complete bibliography and an understandable account of the present state of this subject. Additional material is contained in the book by V.I. Tatarski [2]. The related subject of random wave motion in a non-random medium is surveyed by M. Born and E. Wolf [3, Chap. 10]. Some other work is contained in the symposium volumes edited by W.C. Hoffman [4] and Z. Kopal [5]. Because of the complete bibliographies in these books, we shall give relatively few references.

Physicists have studied extensively the propagation of waves through random collections of discrete scatterers. A clear formulation of a problem of this type, together with a new dishonest method for treating it, was given by L.L. Foldy[6]. This method was generalized to other problems by M. Lax[7], who has also reviewed much of the previous work in this field. Many problems have been treated by this method and others by V. Twersky[8]. Recently, J. Bazer[9] proved that for a one-dimensional scattering problem, this method does yield the correct result. I. Kay and R.A. Silverman[10] investigated the accuracy of another method, the single scattering approximation, by determining the extent to which randomness reduces the importance of multiple scattering.

The next section describes how wave propagation in a random medium applies to the problem of the twinkling of a star. In Section 3 we consider light rays in a slightly inhomogeneous medium, and in Section 4 we apply these results to a random medium obtaining some new results. We then compare these honest results with corresponding dishonest ones in Section 5, where we treat a light ray in a random medium as a Markoff process. In Sections 6 and 7 we use the results of Sections 3 and 4 to determine the phase and amplitude fluctuations of a wave in a random medium. Our results are extensions of previously known ones. This completes our discussion of geometrical optics in a random medium . In the final section we present a brief new treatment of the reduced wave equation in a random medium. This section is independent of the preceding ones.

2. An application

Before illustrating the techniques used to analyze wave propagation in random media, we shall describe a physical phenomenon in which such propagation plays a role. It is the scintillation or twinkling of a star. The most

appropriate theory of wave propagation to describe this phenomenon is geometrical optics. According to this theory, light travels along certain straight or curved paths called rays. These rays are determined by ordinary differential equations in which appears the index of refraction $n(x,t)$, a scalar function which characterizes the transmission medium. This medium is the earth's atmosphere in the present case. If $n=1$ the rays are straight lines which emanate from the star in all directions. One of them (really a narrow beam) enters the eye of an observer who is viewing the star. The direction from which the ray enters the eye is the apparent direction of the star. Actually $n(x,t)$ differs from unity by a small amount. As a consequence, the rays deviate slightly from straight lines and enter the observer's eye from slightly different directions at successive instants of time. Therefore the star appears to be moving about its mean position. Its apparent intensity also fluctuates.

We would like to calculate the apparent direction and intensity as functions of time. To do so it would be necessary to know the index $n(x,t)$. This is practically impossible because the variation of $n(x,t)$ with time and position results from the trubulent motion of the atmosphere. Consequently analysis would appear to be impossible. In the face of this difficulty we treat the atmosphere as a random medium. The random medium must be so chosen that its important statistical properties correspond to measurable properties of the atmosphere. In this way we give up the possibility of calculating the apparent direction and intensity at any particular time. Instead we can calculate statistical properties of the apparent direction and intensity which may be related to the actual temporal distribution of apparent directions and intensities. We shall consider this example further in the next section. In doing so we shall make use of the fact that a ray traverses the atmosphere so

quickly that the index does not change significantly during the traversal. Consequently we may assume that the index is independent of time in determining the rays.

3. Light rays in a slightly inhomogeneous medium

Let $n(x,\epsilon)$ denote the index of refraction of a random medium, which may be written as

$$n(x,\epsilon) = 1 + \epsilon\ \mu(x)\ . \tag{1}$$

The quantity ϵ measures the deviation of the index from unity. The index also depends upon a parameter ω which we shall not write explicitly. We wish to determine the ray $\vec{x}(s,\epsilon)$ which starts from the origin in the direction of the unit vector $\vec{u}$. Here s denotes arclength along the ray. The ray also depends upon ω. The equations which $\vec{x}$ satisfies are

$$(n\vec{x}')' = \nabla n \tag{2}$$

$$\vec{x}(0) = 0 \tag{3}$$

$$\vec{x}'(0) = \vec{u}\ ,\ (\vec{u}^{\,2} = 1)\ . \tag{4}$$

These equations have a unique solution which depends continuously upon n and $\vec{u}$ in an appropriate norm, if n is continuously differentiable. Thus the problem (2) - (4) is well posed.

To find $\vec{x}$ we shall determine its derivatives with respect to ϵ at $\epsilon = 0$ and then express $\vec{x}$ by means of its Taylor series in ϵ. Thus we shall employ an honest method, in the first phase of which probability plays no role. Let us first set $\epsilon = 0$ in (2) - (4) and denote $\vec{x}(s,0)$ by $\vec{x}_o(s)$. Then we obtain

$$\vec{x}_o'' = 0 \tag{5}$$

$$\vec{x}_o(0) = 0 \tag{6}$$

$$\vec{x}_o'(0) = \vec{u}\ . \tag{7}$$

The solution of (5) - (7) is

$$\vec{x}_o(s) = \vec{u}s \ . \tag{8}$$

Now we differentiate (2) - (4) with respect to ϵ, set $\epsilon = 0$, and denote $\vec{x}_\epsilon(s,0)$ by $\vec{x}_\epsilon(s)$, obtaining

$$\vec{x}_\epsilon'' = \nabla \mu(\vec{x}_o) - \left[\vec{x}_o' \cdot \nabla \mu(\vec{x}_o)\right] \vec{x}_o' \tag{9}$$

$$\vec{x}_\epsilon(0) = \vec{x}_\epsilon'(0) = 0 \ . \tag{10}$$

The right side of (9) is just the component of $\nabla\mu$ which is normal to $\vec{x}' = \vec{u}$. Let us call it the transverse gradient and denote it by $\nabla_T \mu$. Then the solution of (9) and (10) is

$$\vec{x}_\epsilon(s) = \int_0^s (s-t)\ \nabla_T\ \mu(\vec{u}t)dt. \tag{11}$$

Differentiating (2) - (4) twice with respect to ϵ at $\epsilon = 0$ yields

$$\vec{x}_{\epsilon\epsilon}''(s) = 2(\vec{x}_\epsilon \cdot \nabla_T)\nabla_T\ \mu(\vec{x}_o) - \nabla_T\ \mu^2(\vec{x}_o) - 2\vec{x}_o'\vec{x}_\epsilon' \cdot \nabla_T\ \mu(\vec{x}_o) - 2\vec{x}_\epsilon'\vec{x}_o' \cdot \nabla\ \mu(\vec{x}_o) \tag{12}$$

$$\vec{x}_{\epsilon\epsilon}(0) = \vec{x}_{\epsilon\epsilon}'(0) = 0 \ . \tag{13}$$

The solution of (12) and (13) is

$$\vec{x}_{\epsilon\epsilon}(s) = 2\int_0^s (s-t)\left[\vec{x}_\epsilon \cdot \nabla_T\ \mu - \frac{1}{2}\nabla_T\ \mu^2 - \vec{u}(\nabla_T\ \mu)^2 - 2(\vec{u}\cdot \nabla\mu)\nabla_T\ \mu\right]dt. \tag{14}$$

In the integrand, the argument of μ is $\vec{u}t$. Thus to the second order in ϵ we have

$$\vec{x}(s,\epsilon) = \vec{x}_o(s) + \epsilon\ \vec{x}_\epsilon(s) + \frac{\epsilon^2}{2}\vec{x}_{\epsilon\epsilon}(s) + O(\epsilon^3) \ . \tag{15}$$

4. Light rays in a random medium

Let us now take account of the randomness and compute some statistical properties of $x(s,\epsilon)$. Let us begin with the mean value $\langle\vec{x}(s,\epsilon)\rangle$ which is just the sum of the mean values of the terms on the right side of (15). Since $\vec{x}_o(s) = \vec{u}s$ is independent of μ and therefore of ω, $\langle\vec{x}_o(s)\rangle = \vec{u}s$. From (11), by interchanging the order of taking the mean value with integration and differentiation, which we assume to be permissible, we obtain

$$\langle\vec{x}_\epsilon(s)\rangle = \int_o^s (s-t)\ \nabla_T \langle\mu(\vec{u}t)\rangle\, dt. \tag{16}$$

In (16) only $\langle\mu\rangle$ occurs. We see that it is not yet necessary to specify the probability space Ω, the measure $dP(\omega)$ nor $\mu(x,\omega)$ if instead we give $\langle\mu(x)\rangle$. We therefore assume that

$$\langle\mu(x)\rangle = 0\ . \tag{17}$$

From (16) and (17) it follows that

$$\langle\vec{x}_\epsilon(s)\rangle = 0\ . \tag{18}$$

To compute the mean value of $\vec{x}_{\epsilon\epsilon}(s)$, which is given by (14), we must know the mean values of certain quadratic expressions in μ and its derivatives. They can be determined from the correlation function $\langle\mu(\vec{x}_1)\mu(\vec{x}_2)\rangle$ which will also be required in later calculations. Therefore we assume that it is given by

$$\langle\mu(\vec{x}_1)\mu(\vec{x}_2)\rangle = \langle\mu^2\rangle\, N(|\vec{x}_1-\vec{x}_2|)\ . \tag{19}$$

Here the mean square fluctuation $\langle\mu^2\rangle$ is assumed to be a constant and the correlation coefficient N is assumed to be a function of the distance $|\vec{x}_1-\vec{x}_2|$ only. By the Schwartz inequality it follows that $|N| \le 1$. Of course $N(0)=1$.

We also require that $N'(0) = 0$ and that $N(\infty) = 0$. It follows that $N_{rr}(0) \leq 0$. In addition N should become small beyond a distance a which we shall call the correlation distance.

From the assumption that $\langle\mu^2\rangle$ is constant we have $\langle\nabla_T \mu^2\rangle = \nabla_T\langle\mu^2\rangle = 0$. Thus the mean of the second term in the integrand in (14) is zero. To compute the mean of the first term we use (11) for $\dot{\vec{x}}_\epsilon$ and the first term becomes

$$\int_0^s\int_0^{t_2} (s-t_2)(t_2-t_1)\left\{\left[\nabla_{T1}\mu(\vec{u}t_1)\cdot\nabla_{T2}\right]\nabla_{T2}\mu(\vec{u}t_2)\right\} dt_1 dt_2 \ . \tag{20}$$

The expression in brackets can be rewritten as $(\nabla_{T1}\cdot\nabla_{T2})\nabla_{T2}\mu(\vec{u}t_2)\mu(\vec{u}t_1)$ and since $\nabla_{T1} = -\nabla_{T2}$, when applied to functions of $t_2 - t_1$ its mean value is

$$\langle\mu^2\rangle(\nabla_{T1}\cdot\nabla_{T2})\nabla_{T2}N(|t_2-t_1|) = -\langle\mu^2\rangle\nabla_{T2}\left[\nabla_{T2}^2 N(|t_2-t_1|)\right] \ . \tag{21}$$

If we set $|t_2-t_1| = r$ then $N = N(r)$ and $\nabla_{T2} = \nabla_T$. Since $\nabla_T^2 N(r)$ is a function of r only, its transverse gradient is zero. Therefore (21) and the mean of (20) are zero. In a similar way we find that the mean of the last term in (14) is also zero. Thus only the third term remains and we obtain

$$\langle\vec{x}(s,\epsilon)\rangle = \vec{u}s - \vec{u}\epsilon^2 \int_0^s (s-t)\langle(\nabla_T \mu)^2\rangle\, dt + O(\epsilon^3) \tag{22}$$

To evaluate $\langle(\nabla_T \mu)^2\rangle$ we have, since $\nabla_{T1} N = -\nabla_{T2} N$ and since $\nabla_T^2 N(0) = \frac{2}{3}\nabla^2 N(0)$,

$$\langle(\nabla_T \mu)^2\rangle = \langle\mu^2\rangle \lim_{x_1\to x_2} \nabla_{T1}\cdot\nabla_{T2} N(|\vec{x}_1-\vec{x}_2|)$$

$$= -\langle\mu^2\rangle\nabla_T^2 N(0) = -\frac{2}{3}\langle\mu^2\rangle\nabla^2 N(0) \ . \tag{23}$$

Thus finally, taking account of the fact that $\nabla^2 N(0) = 3N_{rr}(0)$, w e obtain

$$\langle\vec{x}(s,\epsilon)\rangle = \vec{u}\left(s + \epsilon^2\langle\mu^2\rangle N_{rr}(0)s^2\right) + O(\epsilon^3) \ . \tag{24}$$

From (24) we can solve for the length s such that a ray of this length travels a mean distance L in its original direction. This value of s is

$$s = L - \epsilon^2 \langle\mu^2\rangle N_{rr}(0)L^2 + O(\epsilon^3) \ . \tag{25}$$

Equation (24) gives the mean location of the endpoint of a ray of length s which starts from the origin in the direction $\vec{u}$. This mean position is on the straight line through the origin in this direction, which we expect to be the case from symmetry. However, its distance from the origin is less than s, as we also expect. The terms given in (24) do not suffice to compute $\langle\vec{x}(s,\epsilon)\rangle$ for very large values of ϵs. This can be seen by noting that for such values, the mean position of the endpoint would lie in the direction $-\vec{u}$, according to (24) since $N_{rr}(0) \leq 0$. This result is unreasonable, as is the fact that the mean distance from the origin to the endpoint would exceed s for very large ϵs. We conclude that the approximation (24) for $\langle\vec{x}(s,\epsilon)\rangle$ is not uniform in s. Thus more terms in the series are required for large values of ϵs. They involve third and higher order correlation functions and moments of μ.

Let us now compute the mean value of $\left[\vec{x}(s,\epsilon)\right]^2$. From (15) we have

$$\left[\vec{x}(s,\epsilon)\right]^2 = s^2 + \epsilon^2\left[\vec{x}_\epsilon(s)\right]^2 + \epsilon^2 s\vec{u}\cdot\vec{x}_{\epsilon\epsilon}(s) + O(\epsilon^3) \ . \tag{26}$$

In writing (26) we made use of the fact that $\vec{u}\cdot\vec{x}_\epsilon = 0$. By using (11) and (14) in (26) we obtain

$$\left[\vec{x}(s,\epsilon)\right]^2 = s^2 + \epsilon^2\left(\int_0^s (s-t)\nabla_T\mu dt\right)^2 - 2\epsilon^2 s\int_0^s (s-t)(\nabla_T\mu)^2 dt + O(\epsilon^3) \ . \tag{27}$$

The mean value of the third term on the right side in (27) has been computed in reducing (23) to (24). To compute the mean of the second term we write it as follows

$$<(\vec{x}_\epsilon)^2> = < \int_0^s (s-t_1)\nabla_T\mu(\vec{u}t_1)dt_1 \cdot \int_0^s (s-t_2)\nabla_T(\vec{u}t_2)dt_2 >$$

$$= <\mu^2> \int_0^s\int_0^s (s-t_1)(s-t_2)\nabla_{T1} \cdot \nabla_{T2}\, N(|\vec{u}t_1 - \vec{u}t_2|)dt_1dt_2$$

$$= - <\mu^2> \int_0^s\int_0^s (s-t_1)(s-t_2)\, \nabla_T^2\, N(|t_1-t_2|)dt_1dt_2 \,. \tag{28}$$

We now define r and r_o by

$$t_1 = r_o + r/2 \qquad t_2 = r_o - r/2 \,. \tag{29}$$

Then (28) becomes, if we define $N(-r) = N(r)$,

$$<(\vec{x}_\epsilon)^2> = - <\mu^2> \int_{-s}^{o} \int_{-r/2}^{s+(r/2)} \left[(s-r_o)^2 - \frac{r^2}{4}\right] \nabla_T^2\, N(r)dr_o dr$$

$$- <\mu^2> \int_0^s \int_{r/2}^{s-(r/2)} \left[(s-r_o)^2 - \frac{r^2}{4}\right] \nabla_T^2\, N(r)dr_o dr \,. \tag{30}$$

Upon performing the integration with respect to r_o and combining the resulting integrals we obtain from (30)

$$<(\vec{x}_\epsilon)^2> = - <\mu^2> \int_0^s \nabla_T^2\, N(r)\left[\frac{r^3}{3} - rs^2 + \frac{2s^3}{3}\right]dr \,. \tag{31}$$

To simplify (31) we note that

$$\nabla_T^2\, N(r) = 2r^{-1}\, N_r(r) \,. \tag{32}$$

With the aid of (32), we can rewrite (31) as

$$\epsilon^2 \langle(\vec{x}_\epsilon)^2\rangle = -\epsilon^2 \langle\mu^2\rangle \int_0^s N_r(r)\left[\frac{2r^2}{3} - 2s^2 + \frac{4s^3}{3r}\right]dr \ . \tag{33}$$

Now (27) becomes

$$\langle|\vec{x}(s,\epsilon)|^2\rangle = s^2 - \epsilon^2 \langle\mu^2\rangle \left[\int_0^s N_r(r)\left\{\frac{2r^2}{3} - 2s^2 + \frac{4s^3}{3r}\right\}dr - 2s^3 N_{rr}(0)\right] + O(\epsilon^3). \tag{34}$$

For values of s large compared to the correlation distance a, (34) can be simplified to

$$\langle|\vec{x}(s,\epsilon)|^2\rangle \approx s^2 - \epsilon^2 \langle\mu^2\rangle s^3\left[\frac{4}{3}\int_0^\infty r^{-1} N_r(r)dr - 2N_{rr}(0)\right] + O(\epsilon^3) \ldots (35)$$

Equation (34) or (35) gives the mean square distance between the endpoints of a ray of length s. Let ρ denote the transverse displacement of the endpoint of a ray of length s from the straight line tangent to the ray at its initial point. Then equation (33) gives, to order ϵ^2, the mean square value $\langle\rho^2\rangle$. For s large compared to the correlation distance a, (33) yields for $\langle\rho^2\rangle$

$$\langle\rho^2\rangle \approx -\frac{4\epsilon^2 \langle\mu^2\rangle s^3}{3}\int_0^\infty r^{-1} N_r dr + O(\epsilon^3) \ . \tag{36}$$

By substraction of (36) from (35) we obtain the mean square distance which the ray travels along its original direction. For s large compared to a, it is

$$\langle|\vec{u}\cdot\vec{x}(s,\epsilon)|^2\rangle \approx s^2 + 2\epsilon^2 \langle\mu^2\rangle N_{rr}(0)s^3 + O(\epsilon^3) \ . \tag{37}$$

Let us finally calculate the mean square value of $\vec{x}'(s,\epsilon) - \vec{u}$. This is the mean square value of $2 - 2\cos\alpha(s,\epsilon)$ where α is the angle between $\vec{x}'$ and $\vec{u}$. From (15) we have

$$(\vec{x}' - \vec{u})^2 = \epsilon^2(\vec{x}'_\epsilon)^2 + O(\epsilon^3) \ . \tag{38}$$

From (38) and (11) we obtain

$$<(\vec{x}' - \vec{u})^2> = \epsilon^2 < \left(\int_0^s \nabla_T \mu \, dt\right)^2 > + O(\epsilon^3) \ . \tag{39}$$

By proceeding as in (28) - (33) we find

$$<(\vec{x}' - \vec{u})^2> = 2\Big(1 - <\cos\alpha(s,\epsilon)>\Big) = 4\epsilon^2 <\mu^2> \left[N(s) - 1 - s\int_0^s r^{-1}N_r dr\right] + O(\epsilon^3). \tag{40}$$

For s large compared to a this becomes

$$<(\vec{x}' - \vec{u})^2> = 2\Big(1 - <\cos\alpha(s,\epsilon)>\Big) \approx -4\epsilon^2 <\mu^2> s\int_0^\infty r^{-1}N_r dr + O(\epsilon^3) \ . \tag{41}$$

If the right side of (41) is small compared to unity, we have $2(1 - <\cos\alpha>) \approx <\alpha^2>$. Then (41) gives the mean square value of α.

The results (36) and (41) can be written simply in terms of the ray diffusion coefficient D which is defined by

$$D = -\epsilon^2 <\mu^2> \int_0^\infty r^{-1} N_r(r)dr \ . \tag{42}$$

Then (36) and (41) become

$$<\rho^2> \approx \frac{4}{3} Ds^3 + O(\epsilon^3). \tag{43}$$

$$<\alpha^2> \approx 4Ds + O(\epsilon^3). \tag{44}$$

An incorrect form of (41) was derived by L.A. Chernov [1,p.17,eq.31]. His method of derivation is the same as that given here but he did not introduce a small parameter ϵ and expand systematically in powers of it. As a consequence he made an error in going from his eq. (25) to eq. (26) on page 16. Therefore his result eq. (31) and his diffusion coefficient eq. (34) page 17 are incorrect.

5. Light rays as a Markoff process

In the preceding section we treated light rays in a slightly inhomogeneous random medium by an honest method. We computed the mean values of some quantities associated with the rays and obtained simple expressions for them when the ray length s was large compared to the correlation distance a. Although the results are correct, they are not useful for very large values of s, as can be seen by examining them. We must conclude that the quantities we have calculated in powers of ϵ are not uniform in s.

To obtain results which are valid for large values of s a dishonest method has been employed[1]. It consists in treating the ray direction $\vec{x}'(s)$ as a random function of the arclength s. The fundamental unproved assumption on which the method is based is that $\vec{x}'(s)$ is a continuous Markoff process. This means that the probability that the ray has a given direction at s + ds depends only upon the probability distribution at s. Let us denote this probability density by $P(\theta,\phi,s)$. Then as a consequence of conservation of probability, P satisfies the Chapman-Kolmogoroff equation. The differential form of this equation is the Fokker-Planck equation

$$P_s = \left[\frac{1}{\sin\theta} (\sin\theta\, P_\theta)_\theta + \frac{1}{\sin^2\theta} P_{\phi\phi} \right] \cdot \lim_{\Delta s \to 0} \frac{\langle \alpha^2 \rangle}{4\Delta s} . \tag{45}$$

Here $\langle \alpha^2 \rangle$ is the mean square value of the angle α between $\vec{x}'(s)$ and $\vec{x}'(s+\Delta s)$

when $x'(s)$ is given. We have calculated it in the preceding section and it is given by (40). From (40) it follows at once that $\langle\alpha^2\rangle$ is of order $(\Delta s)^2$ for small Δs and thus the limit in (45) is zero. However, if we use (44) for $\langle\alpha^2\rangle$ instead of (40), and neglect $O(\epsilon^3)$, the limit in (45) is D and (45) becomes

$$P_s = \frac{D}{\sin\theta}(\sin\theta\, P_\theta)_\theta + \frac{D}{\sin^2\theta} P_{\phi\phi} . \tag{46}$$

It would seem to be inconsistent to use (44), which is valid for s large compared to the correlation distance a, and then to let s tend to zero. But this kind of procedure is common in applied mathematics. It can usually be justified by interpreting it as determining an asymptotic expansion with respect to some appropriate parameter. In the present case this is so and the parameter is the correlation distance a. To show this we shall compute the limit of $s^{-1}a\langle\alpha^2\rangle$ as a tends to zero, by means of (40). In doing so we first observe that, from dimensional considerations, N is a function of $a^{-1}r$, say $N(r) = M(a^{-1}r)$. Now (40) yields

$$\lim_{a\to 0} s^{-1}a\,\langle\alpha^2\rangle = \lim_{a\to 0} 4\epsilon^2\,\langle\mu^2\rangle\, s^{-1}\, a\left[M(a^{-1}s)-1-s\int_0^s r^{-1}a^{-1}M'(a^{-1}r)dr\right]$$

$$= 4\epsilon^2\,\langle\mu^2\rangle\, s^{-1}\lim_{a\to 0}\left[aM(a^{-1}s)-a-s\int_0^{a^{-1}s} x^{-1}M'(x)dx\right]$$

$$= 4\epsilon^2\,\langle\mu^2\rangle\int_0^\infty x^{-1}M'(x)dx . \tag{47}$$

From (47) and (42) we see that for any $s \neq 0$, $(4s)^{-1}\langle\alpha^2\rangle \sim D$ as a tends to zero. Then the limit in (45) is D. We now see that (46) is valid in the

limit in which the correlation distance a tends to zero. Thus the solution of (46) is the leading term in the expansion of P for small values of a.

We have seen that the assumption that $\vec{x}'(s)$ is a continuous Markoff process leads to (45). When a tends to zero (45) becomes (46). If P is independent of ϕ, which is the case for all s if it is so for s = 0, (46) further simplifies to

$$P_s = \frac{D}{\sin\theta}(\sin\theta\, P_\theta)_\theta \ . \tag{48}$$

Let us now solve (48) subject to the 'initial' condition

$$P(0,\theta,\phi) = \delta(\theta) \ . \tag{49}$$

This 'initial' condition corresponds to the ray at s = 0 certainly pointing in the direction $\theta = 0$. The solution of (48) and (49) is expressible in terms of Legendre polynomials $P_n(\cos\theta)$ as

$$P(s,\theta,\phi) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)e^{-n(n+1)Ds} \ . \tag{50}$$

From (50) we see that $P(\infty,\theta,\phi) = 1/4\pi$ so that for large s all directions become equally probable. From (50) we easily find

$$\langle\cos\theta\rangle = e^{-2Ds} \ . \tag{51}$$

Let us compare (51) with the result for $\langle\cos\alpha\rangle$ which can be obtained from (40). We observe that α and θ are defined in the same way so that the two results should agree. However, they do not agree. Since (40) was obtained by an honest method and (51) by a dishonest one, we might be tempted to conclude that the dishonest method has yielded an incorrect result. But the preceding discussion in this section shows that this is not the proper conclusion. Instead we must recognize that the two results have different domains

of validity. The result (40) is valid for small values of s while (51) is valid for large values of s. If we expand (41) for s large and (51) for s small we obtain agreement to two terms:

$$\langle \cos \alpha \rangle \approx 1 - 2Ds \approx \langle \cos \theta \rangle . \tag{52}$$

This agreement shows that the two domains of validity overlap slightly. Of course the question of whether the result (51) is valid at all, since it is based on the Markoffian assumption, is not answered by these considerations.

The result (51) can be used to determine the mean square distance between the endpoints of a ray of length s. To use it for this purpose we write

$$\langle |\vec{x}(s)|^2 \rangle = \left\langle \int_0^s \vec{x}'(t_1)dt_1 \cdot \int_0^s \vec{x}'(t_2)dt_2 \right\rangle$$

$$= \int_0^s\int_0^s \langle x'(t_1)\cdot x'(t_2)\rangle \, dt_1 dt_2$$

$$= \int_0^s\int_0^s \langle \cos \theta(t_1,t_2)\rangle \, dt_1 dt_2$$

$$= \int_0^s\int_0^s \exp(-2D|t_1-t_2|) \, dt_1 dt_2$$

$$= \frac{s}{D} - \frac{1}{2D^2}\left(1 - e^{-2Ds}\right) . \tag{55}$$

For small s this becomes

$$\left\langle \left[\vec{x}(s)\right]^2 \right\rangle \approx s^2 - \frac{2}{3} Ds^3 . \tag{54}$$

The first term of this result does agree, but the second does not agree, with (35), which gives the previously calculated value of $\left\langle\left[\vec{x}(s)\right]^2\right\rangle$ for large s. That result involves $N_{rr}(0)$ in addition to D, while all the results of the present method involve only D. We must conclude that the domains of validity of (53) and (34) do not overlap.

Let us finally compute $\left\langle\left(\vec{x}(s)\cdot\vec{u}\right)^2\right\rangle$ by the present method. This is the mean square value of the distance which the ray travels along its initial direction. It is given by

$$\left\langle\left(\vec{x}(s)\cdot\vec{u}\right)^2\right\rangle = \int_0^s\int_0^s \langle\cos\,\theta(t_1)\cos\,\theta(t_2)\rangle\, dt_1 dt_2 \,. \tag{55}$$

By using (50) we finally obtain

$$\begin{aligned}\left\langle\left(\vec{x}(s)\cdot\vec{u}\right)^2\right\rangle &= \frac{s}{3D} - \frac{1}{18D^2}\left(1 - e^{-6Ds}\right)\\ &\approx s^2 - 2Ds^3 \,.\end{aligned} \tag{56}$$

The second form of (56) holds for small Ds. Its first term agrees with (37), which gives the same quantity computed by the previous method. However, the second term does not agree.

By substraction of (56) from (53) we obtain $\langle\rho^2\rangle$, the mean square transverse displacement of the endpoint of the ray from its initial tangent line. It is given by

$$\begin{aligned}\langle\rho^2\rangle &= \frac{2s}{3D} - \frac{1}{D^2}\left(\frac{4}{9} - \frac{e^{-2Ds}}{2} + \frac{e^{-6Ds}}{18}\right)\\ &\approx \frac{4}{3}\,Ds^3 \,.\end{aligned} \tag{57}$$

The second form of (57), which holds for small Ds, agrees with (43), obtained by the previous method.

These calculations suffice to illustrate this method and to show to what extent its results agree, or disagree, with those of the preceding honest method.

6. Phase fluctuations

In geometrical optics rays do not occur individually but only in normal congruences. A normal congruence is a two parameter family of rays, all of which are normal to some smooth surface. Every surface normal to the rays of such a congruence is called a wavefront. With every normal congruence of rays we associate a complex valued field $v(\vec{x})$ which we write as

$$v(\vec{x}) = A(\vec{x})\, e^{ik\phi(\vec{x})}. \tag{58}$$

The possibly complex factor $A(\vec{x})$ is called the amplitude of the field and $\phi(\vec{x})$ is called its phase. These quantities are uniquely defined in terms of v by the requirements that ϕ be independent of the propagation constant k and that A depend upon k at most through a factor of a power of k. The propagation constant k is equal to $2\pi/\lambda$ where λ is the wavelength of the field in a region of refractive index equal to unity. The amplitude $A(\vec{x})$ may be either a scalar or a vector. The real time harmonic field $V(\vec{x},t)$ determined by $v(\vec{x})$ is given by $V(\vec{x},t) = \mathrm{Re}\left[v(\vec{x})e^{-i\nu t}\right]$. Here t denotes the time and ν the angular frequency of the field.

Let $\vec{x} = \vec{z}(\alpha,\beta)$ be the equation of a wavefront S_o, i.e. a surface orthogonal to a normal congruence of rays. We shall suppose that the parameters α and β are so chosen that at α_o,β_o the following equations hold

$$(\vec{z}_\alpha)^2 = (\vec{z}_\beta)^2 = 1,\ \vec{z}_\alpha\cdot\vec{z}_\beta = 0,\ \vec{u}_\alpha = \rho_1^{-1}\,\vec{z}_\alpha,\ \vec{u}_\beta = \rho_2^{-1}\,\vec{z}_\beta,\ \vec{z}_\alpha\times\vec{z}_\beta = \vec{u}. \tag{59}$$

Here $\vec{u}$ denotes the unit normal to S_o while ρ_1 and ρ_2 are the principle radii of S_o at α_o, β_o. These equations imply that at α_o, β_o the parameters α and β measure arclength along the lines of principle curvature on S_o.

We denote by $\vec{x}(\alpha,\beta,s)$ the ray normal to S_o at α,β. Thus measuring s from S_o, we have

$$\vec{x}(\alpha,\beta,0) = \vec{z}(\alpha,\beta) \tag{60}$$

$$\vec{x}'(\alpha,\beta,0) = \vec{u}(\alpha,\beta) \tag{61}$$

The ray through a given point $\vec{y}$ is determined by the values of α, β and s such that

$$\vec{y} = \vec{x}(\alpha,\beta,s) \ . \tag{62}$$

Once these values are determined, we define the phase $\phi(\vec{y})$ by

$$\phi(\vec{y}) = \int_0^s n\left[\vec{x}(\alpha,\beta,t)\right]dt = s + \epsilon \int_0^s \mu\left[\vec{x}(\alpha,\beta,t)\right]dt \tag{63}$$

We have taken ϕ to be zero on S_o.

To evaluate ϕ we must first solve (62) which we shall do by expansion in powers of ϵ. By using (15) we can write (62) as

$$\vec{y} = \vec{x}_o(\alpha,\beta,s) + \epsilon\, \vec{x}_\epsilon(\alpha,\beta,s) + O(\epsilon^2). \tag{64}$$

Here $\vec{x}_o(\alpha,\beta,s)$ is given by

$$\vec{x}_o = \vec{z}(\alpha,\beta) + s\vec{u}(\alpha,\beta) \ . \tag{65}$$

Let us set $\epsilon = 0$ in (64) and denote the solution by α_o, β_o, s_o. Then by differentiating (64) with respect to ϵ and setting $\epsilon = 0$ we obtain

$$0 = \alpha_\epsilon \vec{x}_{o\alpha}(\alpha_o,\beta_o,s_o) + \beta_\epsilon \vec{x}_{o\beta}(\alpha_o,\beta_o,s_o) + s_\epsilon \vec{x}_o'(\alpha_o,\beta_o,s_o) + \vec{x}_\epsilon(\alpha_o,\beta_o,s_o) \ . \tag{66}$$

By using (65) and (59) and the fact that $\vec{u}\cdot\vec{x}_\epsilon = 0$ we readily obtain from (66) the results

$$\alpha_\epsilon = \frac{-\vec{z}_\alpha\cdot\vec{x}_\epsilon(\alpha_o,\beta_o,s_o)}{1 + s_o\rho_1^{-1}} \tag{68}$$

$$\beta_\epsilon = \frac{-\vec{z}_\beta\cdot\vec{x}_\epsilon(\alpha_o,\beta_o,s_o)}{1 + s_o\rho_2^{-1}} \tag{69}$$

$$s_\epsilon = 0 \ . \tag{69}$$

We now use these results in (63). Since $\mu_\epsilon = \nabla\mu\cdot(\vec{x}_\epsilon + \alpha_\epsilon\vec{x}_{o\alpha} + \beta_\epsilon\vec{x}_{o\beta})$ for $\epsilon = 0$, we obtain

$$\begin{aligned}\phi(\vec{y}) = s_o &+ \epsilon \int_o^{s_o} \nabla\mu\left[\vec{x}_o(\alpha_o,\beta_o,t)\right]dt \\ &+ \epsilon^2 \int_o^{s_o} \nabla\mu\left[\vec{x}_o(\alpha_o,\beta_o,t)\right]\cdot\left[\vec{x}_\epsilon(t) - \frac{\vec{z}_\alpha\cdot\vec{x}_\epsilon(s_o)}{1 + s_o\rho_1^{-1}}(1 + t\rho_1^{-1})\vec{z}_\alpha\right. \\ &\left. - \frac{\vec{z}_\beta\cdot\vec{x}_\epsilon(s_o)}{1 + s_o\rho_2^{-1}}(1 + t\rho_2^{-1})\vec{z}_\beta\right]dt + O(\epsilon^3) \ .\end{aligned} \tag{70}$$

We next use (11) for $\vec{x}_\epsilon$ in (70) and compute the mean value of $\phi(\vec{y})$. A lengthy but rather straightforward calculation yields

$$\langle\phi(\vec{y})\rangle = s_o + \epsilon^2\langle\mu^2\rangle\left(\frac{1}{\rho_1+s_o} + \frac{1}{\rho_2+s_o}\right)\left(\frac{s_o^2}{2} + \int_o^{s_o}(r-s_o)N(r)dr\right) + O(\epsilon^3) \ . \tag{71}$$

If ρ_1 and ρ_2 are both finite then, for large s_o, (71) becomes

$$\langle\phi(\vec{y})\rangle \approx s_o(1 + \epsilon^2\langle\mu^2\rangle) + O(\epsilon^3) \ . \tag{72}$$

Thus the mean phase increases linearly with s_o as if the medium had the constant index of refraction $n^* = 1 + \epsilon^2 \langle \mu^2 \rangle$.

The variance $\sigma^2\left[\phi(\vec{y})\right]$ is given by

$$\sigma^2\left[\phi(\vec{y})\right] = \left\langle \left[\phi(\vec{y}) - \langle\phi(\vec{y})\rangle\right]^2 \right\rangle$$

$$= \left\langle \left[s_o + \epsilon\phi_\epsilon + \frac{\epsilon^2}{2}\phi_{\epsilon\epsilon} - \langle s_o + \epsilon\phi_\epsilon + \frac{\epsilon^2}{2}\phi_{\epsilon\epsilon}\rangle\right]^2 \right\rangle + O(\epsilon^3)$$

$$= \epsilon^2 \langle \phi_\epsilon^2 \rangle + O(\epsilon^3) \ . \tag{73}$$

From (70) and (73) we have

$$\sigma^2\left[\phi(\vec{y})\right] = \epsilon^2 \int_0^{s_o}\int_0^{s_o} \left\langle \mu\left[\vec{x}_o(t_1)\right]\mu\left[\vec{x}_o(t_2)\right] \right\rangle dt_1 dt_2 + O(\epsilon^3)$$

$$= 2\epsilon^2 \langle \mu^2 \rangle \int_0^{s_o} (s_o - r)N(r)dr + O(\epsilon^3) \ . \tag{74}$$

For large s_o this becomes

$$\sigma^2\left[\phi(\vec{y})\right] \approx 2\epsilon^2 \langle \mu^2 \rangle s_o \int_0^\infty N(r)dr \ . \tag{75}$$

In concluding our discussion of the phase ϕ, we should observe that ϕ is a solution of the eiconal equation

$$(\nabla\phi)^2 = n^2 \ . \tag{76}$$

It is a solution which satisfies the boundary condition

$$\phi = 0 \qquad \text{on } S_o \ . \tag{77}$$

There are two such solutions which merely differ in sign. Thus the analysis of this section has concerned the solution of a boundary value problem for a

random partial differential equation of first order. The analysis was based upon the solution of the equation by means of the characteristics, which are the rays in the present case.

7. Amplitude fluctuations

The amplitude $A(\vec{x})$ also satisfies a first order partial differential equation in which $\phi(\vec{x})$ occurs. For a scalar A it is

$$\nabla\cdot(A^2\nabla\phi) = 0 \ . \tag{78}$$

Let us seek the solution which has the prescribed value $A^o(\alpha,\beta)$ on the surface S_o given by $\vec{x} = \vec{z}(\alpha,\beta)$. Thus

$$A\left[\vec{z}(\alpha,\beta)\right] = A^o(\alpha,\beta) \ . \tag{79}$$

These equations can be solved for A by the method of characteristics. Because of the special form of (78) the solution can be obtained easily by integrating (78) over a volume. Now the volume integral of the divergence of a vector is equal to the surface integral of the normal component of the vector. To simplify this surface integral we choose as the volume a region bounded by a tube of rays, the wavefront S_o and another wavefront S. Since $\nabla\phi$ is tangent to the rays, the surface integral over the tube vanishes. Thus only the integrals over the two portions of the wavefronts remain and we obtain

$$\int_{S'} A^2|\nabla\phi|d\sigma - \int_{S'_o} A^2|\nabla\phi|d\sigma_o = 0 \ . \tag{80}$$

In (80) S' and S'_o are the two portions of the wavefronts, $d\sigma$ and $d\sigma_o$ are the elements of area, and we have made use of the fact that $\nabla\phi$ is normal to the wavefronts, pointing inward on S_o and outward on S. Since (80) holds for every choice of S'_o, and the corresponding S' determined by the rays, we can conclude that

$$A^2|\nabla\phi| = A^2(0)|\nabla\phi(0)| \frac{d\sigma_o}{d\sigma} . \tag{81}$$

Here $d\sigma_o/d\sigma$ is the ratio of corresponding area elements on S_o and S, and the argument zero denotes s = 0 which corresponds to a point on S_o. Since $|\nabla\phi| = n$, we obtain from (81) and (79) the result

$$A\left[\vec{x}(\alpha,\beta,s)\right] = A^o(\alpha,\beta)\left(\frac{n\left[\vec{z}(\alpha,\beta)\right]}{n\left[\vec{x}(\alpha,\beta,s)\right]}\right)^{1/2}\left(\frac{d\sigma_o}{d\sigma}\right)^{1/2} . \tag{82}$$

To evaluate A we must compute $d\sigma/d\sigma_o$, which is the Jacobian of the mapping of S_o onto S by means of rays. This mapping is given by the function $\vec{x}(\alpha,\beta,s)$ with s having the constant value corresponding to S. If α and β are defined as before at α_o,β_o on S_o, then along the ray through this point we have

$$\frac{d\sigma}{d\sigma_o} = \left[\vec{x}_\alpha(\alpha_o,\beta_o,s) \times \vec{x}_\beta(\alpha_o,\beta_o,s)\right] . \tag{83}$$

If we use (64) for $x(\alpha,\beta,s)$, (83) becomes

$$\begin{aligned}
\frac{d\sigma}{d\sigma_o} &= \left|\left[(1+\rho_1^{-1}s)\vec{z}_\alpha + \epsilon\vec{x}_{\epsilon\alpha}\right] \times \left[(1+\rho_2^{-1}s)\vec{z}_\beta + \epsilon\vec{x}_{\epsilon\beta}\right]\right| + O(\epsilon^2) \\
&= \left|(1+\rho_1^{-1}s)(1+\rho_2^{-1}s)\vec{u} + \epsilon\left[(1+\rho_1^{-1}s)(\vec{z}_\alpha \times \vec{x}_{\epsilon\beta}) + (1+\rho_2^{-1}s)(\vec{x}_{\epsilon\alpha} \times \vec{z}_\beta)\right]\right| \\
&= (1+\rho_1^{-1}s)(1+\rho_2^{-1}s) + \epsilon\left[(1+\rho_1^{-1}s)\vec{u}\cdot(\vec{z}_\alpha \times x_{\epsilon\beta}) + (1+\rho_2^{-1}s)\vec{u}\cdot(\vec{x}_{\epsilon\alpha} \times \vec{z}_\beta)\right] + O(\epsilon^2) \\
&= (1+\rho_1^{-1}s)(1+\rho_2^{-1}s) + \epsilon\left[(1+\rho_1^{-1}s)\vec{z}_\beta\cdot\vec{x}_{\epsilon\beta} + (1+\rho_2^{-1}s)\vec{z}_\alpha\cdot\vec{x}_{\epsilon\alpha}\right] + O(\epsilon^2) .
\end{aligned} \tag{84}$$

Now (82) yields for A the result

$$A\left[\vec{x}(\alpha,\beta,s)\right]=A_o(\alpha,\beta,s)\left\{1-\frac{\epsilon}{2}\left[(1+\rho_2^{-1}s)^{-1}\,\vec{z}_\beta\cdot\vec{x}_{\epsilon\beta}+(1+\rho_1^{-1}s)^{-1}\,\vec{z}_\alpha\cdot\vec{x}_{\epsilon\alpha}\right.\right.$$

$$\left.\left.-\,\mu(\vec{z})+\mu(\vec{x})\right]\right\}+O(\epsilon^2). \tag{85}$$

Here we have introduced A_o which is defined by

$$A_o(\alpha,\beta,s) = A^o(\alpha,\beta)(1+\rho_1^{-1}s)^{-1/2}(1+\rho_2^{-1}s)^{-1/2}\ . \tag{86}$$

To obtain A at a fixed point, say $\vec{y}$, we must insert for α,β and s in (85) the solutions of (62). They are $\alpha_o+\epsilon\alpha_\epsilon+O(\alpha^2)$, $\beta_o+\epsilon\beta_\epsilon+O(\epsilon^2)$ and $s_o+O(\epsilon^2)$. Then (85) yields

$$A(\vec{y}) = A_o(\alpha_o,\beta_o,s_o)\left\{1-\frac{\epsilon}{2}\left[(1+\rho_2^{-1}s)^{-1}\,\vec{z}_\beta\cdot\vec{x}_{\epsilon\beta}+(1+\rho_1^{-1}s)\,\vec{z}_\alpha\cdot\vec{x}_{\epsilon\alpha}\right.\right.$$

$$\left.\left.-\,\mu(\vec{z})+\mu(\vec{x})\right]+\epsilon A_o^{-1}\left[A_{o\alpha}\alpha_\epsilon+A_{o\beta}\beta_\epsilon\right]\right\}+O(\epsilon^2). \tag{87}$$

From (87) we find at once that

$$\langle A(\vec{y})\rangle = A_o(\alpha_o,\beta_o,s_o)+O(\epsilon^2). \tag{88}$$

Next we compute the variance of $A(\vec{y})$.

$$\langle A^2\rangle-\langle A\rangle^2 = \left\langle\left[A(\vec{y})-A_o\right]^2\right\rangle+O(\epsilon^3)\ . \tag{89}$$

We shall evaluate it in the special case in which $A_{o\alpha}=A_{o\beta}=0$. Then after a lengthy calculation, we find for $s \gg a$,

$$\langle A^2\rangle - \langle A\rangle^2 = \frac{\epsilon^2\langle\mu^2\rangle Cs^3 A_o^2}{12}\left\{\frac{s^2}{5}\left(\frac{1}{\rho_1+s}+\frac{1}{\rho_2+s}\right)^2 + 2\left(\frac{\rho_1}{\rho_1+s}+\frac{\rho_2}{\rho_2+s}\right)^2\right.$$

$$\left. + s\left[\frac{\rho_1}{(\rho_1+s)^2}+\frac{\rho_1+\rho_2}{(\rho_1+s)(\rho_2+s)}+\frac{\rho_2}{(\rho_2+s)^2}\right]\right\} . \tag{90}$$

Here

$$C = \frac{1}{4}\int_0^\infty (\nabla_T^2)^2 N(r)dr = \int_0^\infty \left[r^{-2}N_{rr} - r^{-3}N_r\right]dr . \tag{91}$$

When $\rho_1 = \rho_2 = \rho$, (90) simplifies to

$$\langle A^2\rangle - \langle A\rangle^2 = \frac{\epsilon^2\langle\mu^2\rangle Cs^3 A_o^2}{3(\rho+s)^2}\left\{\frac{s^2}{5} + s\rho + 2\rho^2\right\} . \tag{92}$$

If the initial wavefront is a plane, $\rho = \infty$ and (92) becomes

$$\langle A^2\rangle - \langle A\rangle^2 = \frac{2}{3}\,\epsilon^2\,\langle\mu^2\rangle\, Cs^3\, A_o^2 . \tag{93}$$

If the initial wavefront is a point, $\rho = 0$ and (92) yields

$$\langle A^2\rangle - \langle A\rangle^2 = \frac{\epsilon^2}{15}\,\langle\mu^2\rangle\, Cs^3\, A_o^2 . \tag{94}$$

The result (93) is the same as that of Krasilnikov[1] while (94) is similar to, but not identical with, that of P.G. Bergmann[1].

8. The reduced wave equation in a random medium

Let us consider the solution $u(\vec{x})$ of the following problem

$$\Delta u + k^2 n^2(\vec{x})u = -\delta(\vec{x}) \tag{95}$$

$$\lim_{|\vec{x}|\to\infty} |\vec{x}|\left(\frac{\partial u}{\partial|\vec{x}|} - iknu\right) = 0 . \tag{96}$$

Equation (95) is the reduced wave equation with a source term corresponding to a source of unit strength located at the origin $\vec{x} = 0$. Equation (96) is the radiation condition which asserts that the solution u describes outwardly propagating waves. This problem is well posed for a large class of refractive indices $n(\vec{x})$. Let us assume that $n(\vec{x})$ has the form

$$n(\vec{x}) = 1 + \epsilon\mu(\vec{x}) \ . \tag{97}$$

Of course n, μ and therefore u also depend upon the variable ω which ranges over the probability space Ω, but we shall not write it explicitly.

It seems reasonable to suppose that u can be represented as a power series in ϵ

$$u(\vec{x},\epsilon) = u_o(\vec{x}) + \epsilon u_1(\vec{x}) + \epsilon^2 u_2(\vec{x}) + O(\epsilon^3) \ . \tag{98}$$

Upon inserting (97) and (98) into (95) and (96), and equating to zero the coefficient of each power of ϵ, we obtain

$$\Delta u_o + k^2 u_o = - \delta(\vec{x}) \tag{99}$$

$$\lim_{|\vec{x}| \to \infty} |\vec{x}| \left(\frac{\partial u_o}{\partial |\vec{x}|} - iku_o \right) = 0 \tag{100}$$

$$\Delta u_1 + k^2 u_1 = - 2k^2 \mu u_o \tag{101}$$

$$\lim_{|\vec{x}| \to \infty} |\vec{x}| \left(\frac{\partial u_1}{\partial |\vec{x}|} - iku_1 - ik\mu u_o \right) = 0 \tag{102}$$

$$\Delta u_2 + k^2 u_2 = - k^2 \mu^2 u_o - 2k^2 \mu u_1 \tag{103}$$

$$\lim_{|\vec{x}| \to \infty} |\vec{x}| \left(\frac{\partial u_2}{\partial |\vec{x}|} - iku_2 - ik\mu u_1 \right) = 0 \ . \tag{104}$$

The solution of (99) and (100) is

$$u_o(\vec{x}) = \frac{e^{ik|\vec{x}|}}{4\pi|\vec{x}|} \ . \tag{105}$$

To solve (101) and (102) we apply Green's theorem to the interior of a large sphere and assume that the integral over the sphere tends to zero as its radius becomes infinite. Then we obtain

$$u_1(\vec{x}) = \frac{k^2}{2\pi} \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \mu(\vec{x}')u_o(\vec{x}')d\vec{x}' \ . \tag{106}$$

In a similar way we can obtain the solution $u_2(\vec{x})$ of (103) and (104), which is

$$u_2(\vec{x}) = \frac{k^2}{4\pi} \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \left[\mu^2(\vec{x}')u_o(\vec{x}') + 2\mu(\vec{x}')u_1(\vec{x}')\right]d\vec{x}' \ . \tag{107}$$

Let us now compute the mean value of u which requires the calculation of the three mean values $\langle u_o\rangle$, $\langle u_1\rangle$ and $\langle u_2\rangle$. From (105), since u_o is independent of μ and therefore of ω, $\langle u_o\rangle = u_o$. From (106), since u_1 is linear in μ and since we assume that $\langle\mu\rangle = 0$, it follows that $\langle u_1\rangle = 0$. To compute $\langle u_2\rangle$ from (107) we require $\langle\mu^2\rangle$ and $\langle\mu u_1\rangle$. We assume that $\langle\mu^2(\vec{x})\rangle = \langle\mu^2\rangle$ is a constant, and we compute $\langle\mu u_1\rangle$ by using (106) as follows:

$$\langle\mu(\vec{x})u_1(\vec{x})\rangle = \frac{k^2}{2\pi} \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \langle\mu(\vec{x})\mu(\vec{x}')\rangle\, u_o(\vec{x}')d\vec{x}'$$

$$= \frac{k^2\langle\mu^2\rangle}{2\pi} \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} N(|x-x'|)u_o(x')dx'$$

$$= \frac{k^2\langle\mu^2\rangle}{2\pi} \int \frac{e^{ikr}}{r} N(r)u_o(\vec{x}+\vec{r})d\vec{r} \ . \tag{108}$$

In (108) we have introduced the correlation coefficient $N(|\vec{x}-\vec{x}'|) = \langle\mu(\vec{x})\mu(\vec{x}')\rangle/\langle\mu^2\rangle$, which is assumed to be a function of the distance $|\vec{x}-\vec{x}'|$ only.

The angular integration in (108) on a sphere of radius r can be performed explicitly by making use of the following easily-proved mean value theorem which is a consequence of (99)

$$\frac{1}{4\pi}\int_{|\vec{r}|=r} u_o(\vec{x}+\vec{r})ds = u_o(\vec{x})\,\frac{\sin kr}{kr} \qquad 0 < r < |\vec{x}|$$

$$= u_o(\vec{x})\,\sin k|\vec{x}|\,\frac{e^{ik(r-|\vec{x}|)}}{kr} \qquad r > |\vec{x}| \;. \tag{109}$$

When (109) is inserted into (108) it becomes

$$\langle\mu(x)u_1(x)\rangle = u_o(x)2k\langle\mu^2\rangle\left[\int_0^{|\vec{x}|} e^{ikr}N(r)\sin kr\,dr\right.$$

$$\left. + e^{-ik|\vec{x}|}\sin k|\vec{x}|\int_{|\vec{x}|}^{\infty} e^{2ikr}N(r)dr\right]$$

$$= -iu_o(\vec{x})k\langle\mu^2\rangle\left[\int_0^{\infty}(e^{2ikr}-1)N(r)-\int_{|\vec{x}|}^{\infty}(e^{2ik(r-|\vec{x}|)}-1)N(r)dr\right] \;. \tag{110}$$

For $|\vec{x}| \gg a$, which holds at points many correlation lengths from the source, (110) can be simplified to

$$\langle\mu(\vec{x})u_1(\vec{x})\rangle = \beta\langle\mu^2\rangle u_o(\vec{x}) \;. \tag{111}$$

Here we have introduced the constant β which is defined by

$$\beta = -ik\int_0^{\infty}(e^{2ikr}-1)N(r)dr \;. \tag{112}$$

Now when (111) is used in it, (107) yields

$$\langle u_2(\vec{x})\rangle = \frac{k^2}{4\pi} \langle\mu^2\rangle(1+2\beta) \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} u_o(\vec{x}')d\vec{x}' \ . \tag{113}$$

Upon collecting our results we have

$$\langle u(\vec{x})\rangle = u_o(\vec{x}) + \epsilon^2 \langle u_2(\vec{x})\rangle + O(\epsilon^3)$$

$$= u_o(\vec{x}) + \frac{\epsilon^2 k^2 \langle\mu^2\rangle}{4\pi} (1+2\beta) \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} u_o(\vec{x}')d\vec{x}' + O(\epsilon^3). \tag{114}$$

Although (114) gives $\langle u\rangle$ explicitly, it is advantageous to rewrite it as an integral equation for $\langle u\rangle$. Since $\langle u\rangle$ differs from u_o by $O(\epsilon^2)$, the difference between $\epsilon^2\langle u\rangle$ and $\epsilon^2 u_o$ is $O(\epsilon^4)$ and we may write (114) as

$$\langle u(\vec{x})\rangle = u_o(\vec{x}) + \frac{\epsilon^2 k^2 \langle\mu^2\rangle}{4\pi} (1+2\beta) \int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \langle u(\vec{x}')\rangle d\vec{x}' + O(\epsilon^3). \tag{115}$$

If $O(\epsilon^3)$ is omitted, this equation becomes an integral equation for $\langle u(\vec{x})\rangle$. It is equivalent to the following differential equation and radiation condition for $\langle u\rangle$

$$\Delta\langle u\rangle + k^2(n^*)^2\langle u\rangle = -\delta(\vec{x}) \tag{116}$$

$$\lim_{|\vec{x}| \to \infty} |\vec{x}|\left(\frac{\partial\langle u\rangle}{\partial|\vec{x}|} - ikn^*u\right) = 0 \ . \tag{117}$$

Here the effective index n^* is defined by

$$(n^*)^2 = 1 + \epsilon^2 \langle\mu^2\rangle(1 + 2\beta)$$

$$= 1 + \epsilon^2 \langle\mu^2\rangle \left(1 - 2ik \int_0^\infty (e^{2ikr}-1)N(r)dr\right) \ . \tag{118}$$

Our results (116) and (117) show that up to and including terms of order ϵ^2, the mean value of u propagates as if in a medium of complex refractive index n* given by (118). This is the main result of this section. The imaginary part of kn* is the attenuation coefficient α for a wave in this medium. From (118) it is given by

$$\alpha = 2\epsilon^2 \langle\mu^2\rangle k^2 \int_0^\infty (1 - \cos 2kr)N(r)dr \ . \tag{119}$$

This result agrees with that obtained previously by another method. [1]

The solution of (116) and (117) is

$$\langle u(\vec{x}\rangle = \frac{e^{ikn^*|\vec{x}|}}{4\pi|\vec{x}|} \ . \tag{120}$$

By comparing (120) with (105) we see that $\langle u(\vec{x})\rangle$ is given by replacing k by kn* in $u_o(\vec{x})$. This same rule applies to all pairs of solutions u_o and $\langle u\rangle$ of problems which differ from (95), (96) or (116), (117) merely in the source term. From (120) we have

$$|\langle u(\vec{x})\rangle| = \frac{e^{-\alpha|\vec{x}|}}{4\pi|\vec{x}|} \ . \tag{121}$$

The method which we have used to derive (116) and (117) can also be used to determine $\langle u^2(\vec{x})\rangle$ and from it the variance $\langle u^2(\vec{x})\rangle - \langle u(\vec{x})\rangle^2$. The quantity $\langle u\rangle$ is often called the coherent wave and $\langle u\rangle^2$ the coherent intensity while the variance is called the incoherent intensity. Since it should be clear how to obtain an equation for $\langle u^2\rangle$ which is correct through terms in ϵ^2, we shall not carry out this calculation.

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